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# Structural changes in economic growth models ⋆

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**Abstract:** The paper is devoted to analysis of one-sector growth models and corresponding control problems on optimal distribution of investments. The paper considers a model with a linear production function, which takes into account the feasibility of structural changes in an economy. By introducing dummy variables one can statistically indicate a period when the model undergoes changes. This provides the possibility to switch the model in different modes for providing more accurate forecasts of economic development. For the optimal control problems, the qualitative analysis of the Hamiltonian systems is implemented and solutions are constructed for each model mode. Continuous gluing of the obtained trajectories is obtained as a solution of the optimal control problem with different model modes on the infinite time interval. Comparison of the resulting model trajectory with statistical data reveals that the simulated trends provide sufficiently accurate matching with the real data. Adaptation of model parameters to a new economic mode can be considered as a *learning process* for the entire optimal control model. It makes the model more flexible with respect to the qualitative changes influencing forecasts of economic development.



*Keywords:* optimal control problem, economic growth, Hamiltonian trajectories, econometric data analysis, dummy variables, switching model modes

## 1. INTRODUCTION

Nowadays, investigation of economic processes and phenomena is one of the most significant problems which demands an interdisciplinary approach to the complex systems analysis. In this connection. it is worth to mention fundamental works of the famous economists and mathematicians proposing the methodology for constructing mathematical models which describe interconnection of the most significant macroeconomic factors, in particular, works by K. Arrow, L.V. Kantorovich, R. Sollow, K. Shell, G.M. Grossman, E. Helpman, R. Ayres, M. Intriligator, L. Krushvits and U.Ph. Sharp [Arrow (1985); Grossman and Helpman (1991); Solow (1970); Shell (1969); Ayres et al. (2009)].

Mathematical design is a basis for the statement of an optimal control problem, which is aimed for dynamic optimization of investments in increasing effectiveness of basic production factors [Crespo Cuaresma et al. (2010, 2013); Krasovskii and Tarasyev (2008b); Tarasyev and Watanabe (2001); Sanderson et al. (2010)].

It should be noted that, typically, an economic growth model has the scalability property. Therefore, it can be applied for investigation of macroeconomic aggregate factors and also for analysis of regional processes of the economic development. Moreover, a number of factors influencing dynamics of economic development of a country (or a region) is determined by their significance and it can be increased, when required, as it is proposed in the papers Tarasyev and Watanabe (2001); Krasovskii and Tarasyev (2008b); Tarasyev and Usova (2010). Significance of any factor can vary over time periods, and this fact leads, in particular, to structural changes in the model.

In this paper, the model identification is performed using methods of econometrics [Ayvazyan (2010)]. The identification procedure is focused on production functions of the linear type with switching modes. These functions include dummy variables for revealing possible qualitative changes in the economy of a country. Detection of the structural changes and their treatment in dynamic optimization methods is one of the goals of this research. Based on calibrated models, we consider control problems on optimal distribution of investments in the capital stock of country's economy. The quality of the control process is estimated by the integral consumption index discounted on the infinite time interval [Aseev and Kryazhimskiy (2007); Krasovskii and Tarasyev (2008a); Tarasyev and Usova (2010)]. The problem analysis provided in the paper is based on the Pontryagin maximum principle [Pontryagin et al. (1962)] for the problems with the infinite time horizon [Aseev and Kryazhimskiy (2007); Krasovskii et al. (2008)]. Qualitative analysis of the Hamiltonian systems within the maximum

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principle includes searching stationary regimes and the optimal investment levels which are capable to bring the economy in the domain of the favorable macroeconomic development.

The paper includes numerical results for forecast scenarios of development of the Russian economy, constructed based on the solution of the optimal control problem with linear production functions in the presence of structural changes. In conclusion, the comparison of statistical trends, econometric forecasts and simulated trajectories is provided.

The next section of the paper is devoted to the description of the growth model. Further, the econometric data analysis with dummy variables is performed for identifying model parameters. In the third section, we formulate the optimal control problem for the model of economic growth with switching modes generated by dummy variables and provide its analysis within the Pontryagin maximum principle. Numerical solutions and their comparison with statistic and econometric forecasts are carried out in the last section.

## 2. GROWTH MODEL

### 2.1 Production function

The model operates with two production factors: capital stock $K = K(t)$ and labor $L = L(t)$, which determine the output $Y = Y(t)$ by means of a production function $Y = F(K, L)$. A *production function* (or *output function*) is a functional relationship between the output $Y$ and production factors such as capital $K$, labor $L$ and *etc*.

In the model, it is assumed that production $Y$ depends on capital $K$ and labor $L$ in a linear way

$$Y(t) = \alpha K(t) + \beta L(t), \quad \alpha > 0.$$

The homogeneity property of the production function allows to introduce new relative variables, which are the capital $k(t) = K(t)/L(t)$ and the production $y(t) = Y(t)/L(t)$ per one labor unit. In new variables the production function has the form

$$y(t) = \alpha k(t) + \beta = f(k(t)). \tag{1}$$

### 2.2 Model dynamics

The dynamics of capital $K$ is derived using classical approaches proposed in Solow (1970)

$$\dot{K}(t) = S(t) - \mu K(t), \tag{2}$$

where the function $S(t)$ determines investments in capital $K(t)$, positive scalar $\mu$ is a depreciation rate of the capital stock. Investment $S(t)$ is a part of the output and can be represented as $S(t) = u(t)Y(t)$, where function $u(t)$ is an output share invested in capital. In the model, the parameter $u = u(t)$ plays role of a *control variable.*

The investment share $u = u(t)$ satisfies restrictions

$$0 \leqslant S(t) < Y(t) \;\Rightarrow\; 0 \leqslant \frac{S(t)}{Y(t)} < 1 \;\Rightarrow\; 0 \leqslant u(t) < 1.$$

Assumed that there exists a constant parameter $\overline{u} < 1$ determining the maximum investment level [1], *i.e.*

$$0 \leqslant u(t) \leqslant \overline{u} < 1, \quad u \in \mathcal{U} = [0, \overline{u}]. \tag{3}$$

[1] In numerical experiments maximum investment $\overline{u}$ level is estimated by the value of 0.43 according to the used data set.

It is supposed that the labor dynamics satisfies an exponential growth law

$$\dot{L}(t) = nL(t), \quad n > 0, \tag{4}$$

where nonnegative parameter $n$ is a growth rate of the labor.

**Remark**: Assumption on exponential growth of the labor takes place according to the data of the Russian economy for the period from 1990 to 2013 years (see FSSS (2015)). The estimate value of the parameter $n$ from the data is $n^* = 0.0015$.

Using dynamics of capital $K$ (2), labor $L$ (4) and production function (1), one can derive dynamics of relative capital $k = k(t)$

$$\dot{k}(t) = u(t)f(k(t)) - \delta k(t), \quad k(0) = k_0 = K(0)/L(0). \tag{5}$$

where positive constant $\delta = \mu + n$ denotes the level of capital depreciation rate caused by the capital amortization and growth of the labor force [2].

### 2.3 Balance equation

Under the assumption on the closedness of the economic system, when output $Y(t)$ can be spent on investments $S(t)$ and consumption $C(t)$, the *balance equation* can be represented in the form

$$Y(t) = S(t) + C(t) = u(t)Y(t) + C(t). \tag{6}$$

From the balance equation one can derive the consumption $C(t)$ per one worker

$$c = \frac{C}{L} = (1-u)\frac{Y}{L} = (1-u)y = (1-u)f(k). \tag{7}$$

### 2.4 Quality of control process

The quality of the control process is estimated by the integrated consumption index of the logarithmic type. The utility theory postulates that a logarithmic function determines relative growth of a factor (in this case, consumption) in a time period

$$J(\cdot) = \int_0^{+\infty} e^{-\rho t} \ln c(t) dt, \tag{8}$$

where relative consumption level $c = c(t)$ can be found by the formula (7). Discount factor $\rho$ is estimated by the value of 0.11 in the analyzed data.

## 3. ECONOMETRIC ANALYSIS

Statistical data on macroeconomic indicators of the Russian economy is chosen for the period from 1991 to 2013 FSSS (2015) (see Table 1). In general, independent variables in regression models have continuous domains. However, statistical methods for the model identification do not impose restrictions on regressors' behavior. Specifically, some variables can be discrete. Dummy (or discrete) variables describe qualitative features of the model.

Based on statistical data of the Russian economy for the period from 1991 to 2013, it is shown that in 1997 trends of macroeconomic indicators undergo changes. For revealing

[2] Basing on the data parameter $\delta$ is taken at the level of 14.185% for the model analysis.

| Year, t | $\mathbf{y(t) = Y(t)/L(t)}$ | $\mathbf{k(t) = K(t)/L(t)}$ |
|---|---|---|
| 1991 | 0.021 | 0.031 |
| 1992 | 0.264 | 0.601 |
| 1993 | 2.425 | 0.903 |
| 1994 | 8.972 | 17.945 |
| 1995 | 21.536 | 80.001 |
| 1996 | 30.538 | 201.530 |
| 1997 | 36.276 | 207.823 |
| 1998 | 41.292 | 224.197 |
| 1999 | 75.229 | 223.583 |
| 2000 | 113.235 | 270.691 |
| 2001 | 137.636 | 330.798 |
| 2002 | 165.165 | 401.581 |
| 2003 | 200.189 | 487.629 |
| 2004 | 256.407 | 525.151 |
| 2005 | 323.538 | 621.236 |
| 2006 | 400.709 | 706.963 |
| 2007 | 488.797 | 887.862 |
| 2008 | 602.810 | 1087.583 |
| 2009 | 575.237 | 1219.972 |
| 2010 | 685.271 | 1378.955 |
| 2011 | 826.365 | 1594.656 |
| 2012 | 915.407 | 1784.206 |
| 2013 | 983.127 | 1961.036 |

Table 1. Russian macroeconomic indicators

discrete shifts, a binary variable $r = r(t) \in \{0, 1\}$ is included into the regression model. Parameter $r$ takes zero value in the period $t \in [0, \widetilde{T}]$ before changes occur and unit value after the barrier time $\widetilde{T}$.

The identification of parameters $\alpha$ and $\beta$ of the production function (1) is performed according to the statistical data (see Table 1). The regression equation with the dummy variable $r \in \{0, 1\}$, included into the model for revealing a structural shift in the economic development, has the form $y(t) = \alpha k(t) + \beta + \alpha_d r(k(t) - \widetilde{k})$, where parameter $\widetilde{k}$ is determined by the value of the relative capital $k$ in 1997. The identification procedure provides the following estimations

$$\alpha = 0.329, \quad \alpha_d = 0.213, \quad \beta = 0.178. \tag{9}$$

Standard errors for parameters $\alpha$ and $\alpha_d$ are the following $\sigma_\alpha = 0.089$ and $\sigma_{\alpha_d} = 0.096$ respectively. Estimated values $\widehat{y}$ of the output $y$ are found by the formula

$$\widehat{y} = \alpha^* k + \beta^* + \alpha_d^* r(k - \widetilde{k}). \tag{10}$$

Results of the econometric analysis are depicted on Fig. 1.

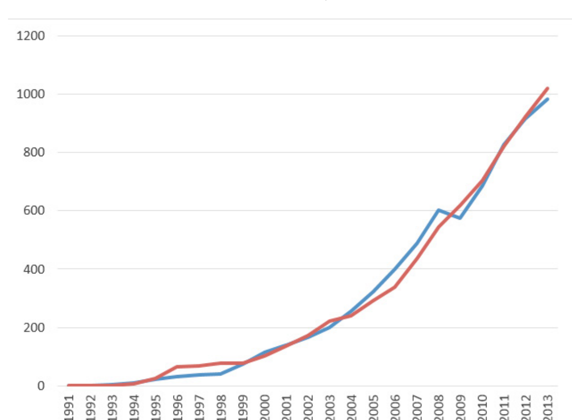


Fig. 1. Real data $y$ and econometric forecast $\widehat{y}$ of the output $y(t)$

The determination coefficient $R^2$ equals to $R^2 = 0.992$, this high value proves a good fitness of the regression model. Significance of the dummy variable $r$ can be shown the using Student criterion. The observable value of the statistics is $\tau_o = \alpha_d^*/\sigma(\alpha_d) = 2.224$, while the critical value of the statistics is $\tau_c = T_{0.05}(20) = 2.086$, which is lesser than the observable value $\tau_o$, consequently, $\alpha_d \neq 0$ with probability of 95 %.

*Remark.* According to the econometric analysis at the considered period of time, Russian economy satisfies the linear regression model quite well (the determination coefficient $R^2$ exceeds 99%).

## 4. CONTROL PROBLEM

**Problem.** The problem consists in searching such investment strategy $(k(t), u(t))$, that satisfies the restrictions on control $u = u(t)$ (3) and maximizes the utility function (8) along trajectories of the system (5) over the infinite time interval.

*Remark.* The optimal control problem is posed at the infinite time horizon, because it aims for longterm predicting of the future development of Russian economy under an assumption that the economic situation does not undergo significant changes. It is supposed that the model parameters stay adequate for the forecasting period. If the economy essentially changes, it leads to the necessity of the additional econometric analysis for evaluating model parameters which correspond to the new economic reality.

The formulated optimal control problem is studied within the framework of the Pontryagin maximum principle [Pontryagin et al. (1962)] for the problems on the infinite time interval [Aseev and Kryazhimskiy (2007)]. There is a number of researches devoted to this problem (see Krasovskii et al. (2008); Krasovskii and Tarasyev (2008b); Tarasyev and Usova (2010, 2011, 2012)), which prove the existence of the unique optimal solution having the property of growth saturation.

### *4.1 Analysis of optimal control problem*

Analysis of the optimal control problem is carried out within the Pontryagin maximum principle [Pontryagin et al. (1962)] generalized for the problems with infinite time interval [Aseev and Kryazhimskiy (2007); Krasovskii et al. (2008)].

According to the maximum principle the stationary Hamiltonian function of the problem has the following form

$$H(k, \psi, u) = \ln f(k) + \ln(1 - u) + \psi(uf(k) - \delta k), \tag{11}$$

where symbol $\psi$ stands for an adjoint variable.

*Proposition 1.* The maximum value of the Hamiltonian function $H(\cdot, \cdot, u)$ (11) with respect to the control parameter $u$ is attained at the point $u^0$

$$u^0(k, \psi) = \begin{cases} 0, & (k, \psi) \in \Delta_1 \\ 1 - \dfrac{1}{f(k)\psi}, & (k, \psi) \in \Delta_2 \\ \overline{u}, & (k, \psi) \in \Delta_3. \end{cases} \tag{12}$$

Domains $\Delta_i$ are determined in the following way: $\Delta_1 = \{(k, \psi) : 0 < f(k)\psi \leqslant 1\}$, $\Delta_2 = \{(k, \psi) : 1 \leqslant f(k)\psi \leqslant \overline{U}\}$ and $\Delta_3 = \{(k, \psi) : f(k)\psi \geqslant \overline{U}\}$, where $\overline{U} = 1/(1 - \overline{u})$.

The validity of the proposition follows from the strictly concavity of the Hamiltonian function $H(\cdot,\cdot,u)$ (11) with respect to the variable $u$ $\left(H''_{u^2} = -(1-u)^{-2} < 0\right)$ and compactness of the set $\mathcal{U} = [0,\overline{u}]$ $(u \in \mathcal{U})$.

Control parameter $u^0(k,\psi)$ (12) splits the domain of variables $(k,\psi)$ into three parts $\Delta_i$, $(i=1,2,3)$. In each domain $\Delta_i$ the Hamiltonian system has different structure determined by the formula

$$\begin{aligned} \dot{k}(t) &= u^0(t)f(k(t)) - \delta k(t), \\ \dot{\psi}(t) &= (\rho+\delta)\psi(t) - \frac{f'(k(t))}{f(k(t))} - u^0(t)f'(k(t))\psi(t). \end{aligned} \tag{13}$$

It is worth to mention, that the construction of forecast trajectories as a solution of the optimal control problem, and econometric data analysis of the production function and other model parameters are solved in two absolutely different model blocks. Namely, forecast trajectories are constructed in the dynamic optimization block, and the identification procedure for model parameters is implemented in econometric block. The econometric data analysis is based on classical method of econometrics Ayvazyan (2010) and is aimed for the identification of model parameters, specifically, elasticity coefficients of the production function (1). Computational algorithms for solving the optimal control problem are executed autonomously using econometrically identified values of the model parameters. The provided solution is an optimal trajectory of endogenous economic growth, which reveals disproportions in the real data with respect to the optimization criterion (8). In this sense, simulated trajectories can be called as optimal forecast trajectories, and they should not necessarily coincide with standard econometric forecasts.

### 4.2 Hamiltonian system

Consider the Hamiltonian system (13) for the linear production function $f(k) = \alpha^* k + \beta^* + \alpha_d^* r(t)(k - \widetilde{k})$ with different control regimes $\Delta_i$ $(i=1,2,3)$ (there regimes are defined in Proposition 1)

$$\begin{aligned} &\Delta_1: \begin{cases} \dot{k}(t) &= -\delta k(t), \\ \dot{\psi}(t) &= (\delta+\rho)\psi(t) - \dfrac{\alpha_i}{\alpha_i k(t) + \beta_i} \end{cases} \\ &\Delta_2: \begin{cases} \dot{k}(t) &= (\alpha_i - \delta)k(t) + \beta_i - \dfrac{1}{\psi(t)}, \\ \dot{\psi}(t) &= -(\alpha_i - (\delta+\rho))\psi(t) \end{cases} \\ &\Delta_3: \begin{cases} \dot{k}(t) &= (\alpha_i\overline{u} - \delta)k(t) + \beta_i\overline{u}, \\ \dot{\psi}(t) &= (\delta+\rho-\alpha_i\overline{u})\psi(t) - \dfrac{\alpha_i}{\alpha_i k(t) + \beta_i} \end{cases} \end{aligned} \tag{14}$$

Here symbols $\alpha_i$ and $\beta_i$ $(i=1,2)$ correspond to the estimation of the production function parameters (10) and take the following values (9) [3]:

$$\alpha_1 = \alpha^*,\ \beta_1 = \beta^*, \quad \alpha_2 = \alpha^* + \alpha_d,\ \beta_2 = \beta^* - \alpha_d^*\widetilde{k}.$$

The initial value of the phase variable $k$ is taken at the level of $k(0) = k_0 = 0.031$.

Analyzing the Hamiltonian system (14), we consider vector fields of the Hamiltonian dynamics for the both branches of the production function (10) corresponding to the periods before and after structural changes.

[3] According to the regression (9)-(10): $\alpha_2 = 0.542$ and $\beta_2 = -44.18$

The vector field of the Hamiltonian system for the production function $f_1(k) = \alpha_1 k + \beta_1$ is depicted on Fig. 2 Red curves $\omega_1 = \omega_1(k)$ and $\omega_2 = \omega_2(k)$ are determined by the equations

$$\psi = \frac{1}{f_1(k) - \delta k} = \omega_1, \quad \psi = \frac{\alpha_1}{(\delta+\rho-\alpha_1\overline{u})f_1(k)} = \omega_2.$$

According to the estimated model parameters and the

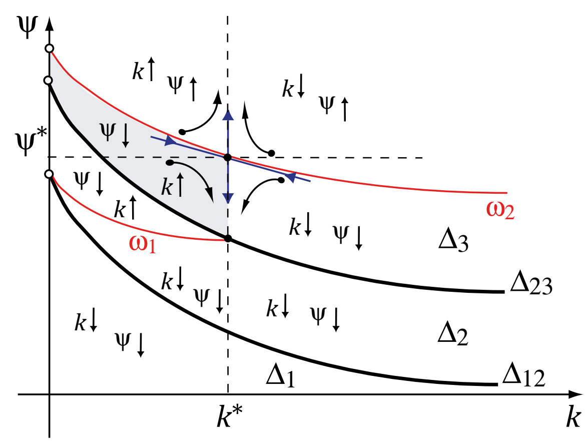


Fig. 2. The vector field of the Hamiltonian system before structural changes

dynamics (14), the trajectory of the phase variables has growth trend in domains $\Delta_2$ and $\Delta_3$. Moreover, the Hamiltonian system in the third domain has a steady state position $(k^*,\psi^*)$ with positive coordinates:

$$k^* = \frac{\beta_1\overline{u}}{\delta - \alpha_1\overline{u}}, \quad \psi^* = \frac{\alpha_1}{(\delta+\rho-\alpha_1\overline{u})f_1(k^*)}. \tag{15}$$

Using the identified model parameters, the steady state value of the capital is estimated at the level of 220.5. The investment level $u(t)$ at the steady state position is determined as $u^* = 0.665$, which is greater than the maximum investment level $\overline{u} = 0.43$. This circumstance evidences that the optimal trajectory of the capital $k$ starts from the point $k_0$ located in the third domain below the stationary level $k^*$ (since $k_0 = 0.031 < 220.5 = k^*$). It implies that the phase variable has the following optimal trajectory

$$k(t) = k^* - (k^* - k_0)e^{-(\delta-\alpha_1\overline{u})t}. \tag{16}$$

Due to the positiveness of the adjoint variable $\psi$ and the transversality condition $\lim\limits_{t\to\infty} e^{-\rho t}\psi(t)k(t) = 0$ (see Aseev and Kryazhimskiy (2007)) the following proposition takes place.

*Proposition 2.* The initial position $(k_0,\psi_0)$ for the optimal solution is located at the domain (gray zone on Fig. 2) bounded from above by the curve $\omega_2$.

**Proof.** In the domain $\Delta_3$ the general solution of the differential equation for the adjoint variable $\psi$ can be represented using the power series

$$\psi(t) = e^{\gamma t}M + \frac{1}{y^*}\sum_{n=0}^{\infty}\frac{\nu^n e^{-n(\delta-\alpha_1\overline{u})t}}{(\delta-\alpha_1\overline{u})n + \gamma} = e^{\gamma t}M + \Psi(t),$$

where $\gamma = \delta + \rho - \alpha_1\overline{u} > 0$, $\nu = (k^* - k_0)/(k^* + \beta_1/\alpha_1) = 0.9972$, $y^* = \alpha_1 k^* + \beta_1$, and the parameter $M$

is a constant determined by means of the transversality conditions

$$0 = \lim_{t\to\infty} e^{-\rho t}k(t)\psi(t) = M + \lim_{t\to\infty} e^{-\rho t}k(t)\Psi(t).$$

The second term tends to zero as the time goes to infinity, while the first one is a constant. Consequently, the transversality condition provides that $M \equiv 0$.

It implies that the joint variable $\psi(t)$ at the domain $\Delta_3$ can be found by the formula

$$\psi(t) = \Psi(t) = \frac{1}{y^*}\sum_{n=0}^{\infty}\frac{\nu^n e^{-n(\delta-\alpha_1\overline{u})t}}{(\delta-\alpha_1\overline{u})n+\gamma}.$$

As a result, one can find the initial value $\psi_0$ of the adjoint variable as follows

$$\psi_0 = \psi(t=0) = \frac{1}{y^*}\sum_{n=0}^{\infty}\frac{\nu^n}{(\delta-\alpha_1\overline{u})n+\gamma} = 9.391.$$

Finally, we determine the location of the initial position $(k_0, \psi_0)$. The value of the variable $\omega_2(k_0)$ is 15.821, and it is larger than the initial value $\psi_0$ of the adjoint variable . Hence, the initial point $(k_0, \psi_0)$ is located below the curve $\omega_2(k)$ in the domain $\Delta_3$. •

Let us consider the Hamiltonian system (14) corresponding to the second branch of the production function $f_2(k) = \alpha_2 k + \beta_2$ (10). In this case, the initial position for the phase variable $k$ is $\widetilde{k} = 207.98$, and the adjoint variable has the value $\widetilde{\psi} = 0.0435$. The initial position corresponds to the domain of the maximum investment level, when $u(t) = \overline{u} = 0.43$, since

$$\widetilde{\psi} = 0.0435 > \Big((1-\overline{u})(\alpha_2\widetilde{k}+\beta_2)\Big)^{-1} = 0.0256$$

The vector field of the Hamiltonian system looks like as it is depicted on Fig. 3. Symbols $\omega_1$ and $\omega_2$ on Fig. 3 indicate

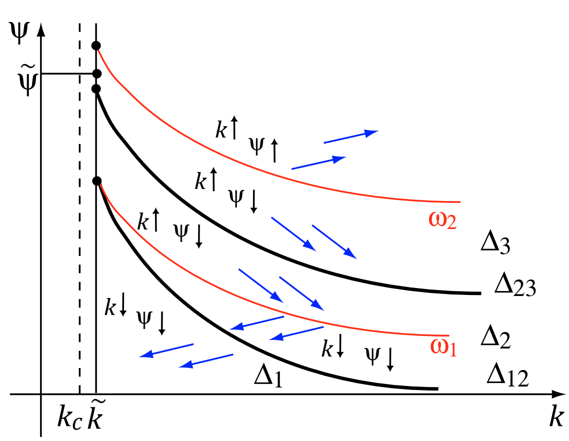


Fig. 3. The vector field of the Hamiltonian system after structural changes

the following curves

$$\psi = \frac{1}{f_2(k)-\delta k} = \omega_1, \quad \psi = \frac{\alpha_2}{(\delta+\rho-\alpha_2\overline{u})f_2(k)} = \omega_2$$

The symbol $k_c$ on the vector field denotes the critical value of the capital, when the right-hand side of the capital dynamic equation is equal to zero, *i.e.*

$$k_c = |\beta_2|\overline{u}/\left(\alpha_2\overline{u}-\delta\right) = 207.93.$$

The value of the parameter $k_c$ does not exceed the value of the initial position $\widetilde{k} = 207.98$ of the capital for the second branch of the production function. Moreover, it implies that the capital $k(t)$ grows over time. The phase trajectory $k = k(t)$ of the Hamiltonian system is described by the relation

$$k(t) = \left(\widetilde{k} + \frac{\beta_2\overline{u}}{\alpha_2\overline{u}-\delta}\right)e^{(\alpha_2\overline{u}-\delta)(t-\widetilde{T})} - \frac{\beta_2\overline{u}}{\alpha_2\overline{u}-\delta}$$

and demonstrates the exponentially growth trend with the positive rate $(\alpha_2\overline{u}-\delta > 0)$. In the next section, the comparison of the obtained optimal solutions with the statistical data is carried out.

## 5. SIMULATED AND STATISTICAL TRENDS

Solutions of the optimal control problem are depicted on Figures 4 and 5 in green color. The trajectory of the output $y(t)$ switches at the point $\widetilde{y} = \alpha_1\widetilde{k} + \beta_1 \approx 68.621$, which divides the solution in periods before and after structural changes of the economy. The real value of the output, corresponding to the switching barrier moment is 36.276, while the econometric forecast provides the value of 68.780, which is very close to the value 68.621 belonging to the optimal trajectory. In the first period, the optimal trajectory of the output $y(t)$ grows and exceeds statistical and econometric trends. However, in the second period, the optimal output approaches the statistical and econometric trends. It is worth to mention that the both optimal and econometric trajectories in the neighborhood of the switching barrier moment of time $\widetilde{T}$, have the saturation level, which is close to value at the steady state $y^* \approx 72.74$ of the output.

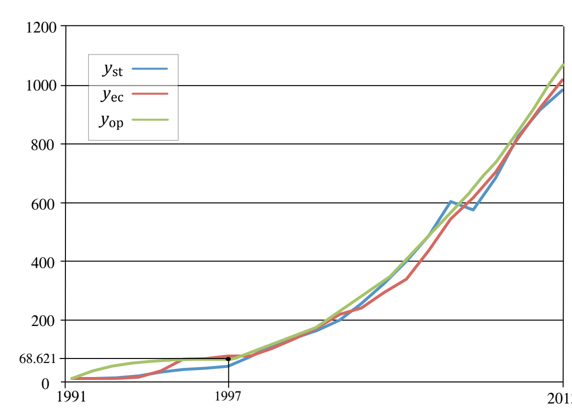


Fig. 4. Statistical, econometric and simulated trends of the production output $y(t)$

The trajectory of the capital $k(t)$ is located higher than the statistical trend for this indicator. In the period $[0, \widetilde{T}]$ before switching, the capital grows and has the saturation level at the value close to the steady state $k^*$. It is interesting to note, that the statistical data in the period around the switching barrier time $\widetilde{T}$ stays almost at the same level. In the second period, the optimal trend of the capital grows exponentially overtakes the statistical data. Nevertheless, both trajectories have the similar behavior, and the optimal solution reflects significant features of the real trends, such as switching between different growth rates, and approaching the local saturation level around the steady state value $k^*$ of the capital $k(t)$.

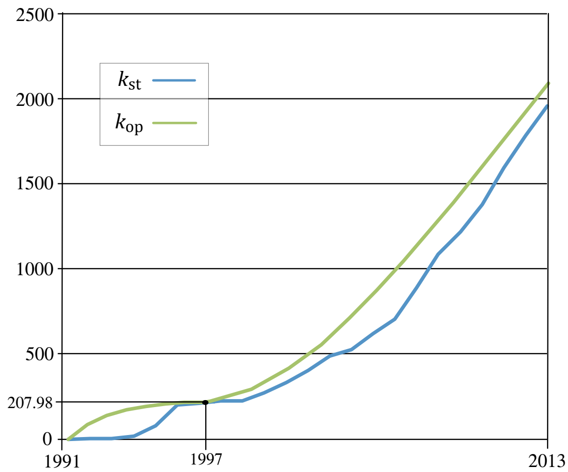


Fig. 5. Statistical and simulated trends of the capital $k(t)$

## 6. CONCLUSION

In the paper, the optimal control problem is solved for the model of economic growth with the production function of the linear type in the presence of switching modes for basic model parameters. The switching modes are detected by means of the classical econometric regression with dummy variables. The optimal control problem for the identified hybrid system is analyzed within the framework of the Pontryagin maximum principle adapted to problems with the infinite time horizon.

Several branches of the Hamiltonian system are constructed, and their qualitative properties are investigated for different switching modes of the model. This analysis includes the construction of steady states for branches of the Hamiltonian system and the indication of growth and decline trends of optimal trajectories. Accurate continuous gluing of elements for the optimal trajectory is implemented in line with the examined properties of the Hamiltonian branches. From the methodological point of view, it is important to note that the econometric analysis and construction of optimal trajectories for the model of economic growth are performed in parallel model blocks which do not intersect, and generate different forecasts. Numerical results are represented graphically, and include simulated trajectories, econometric and statistics trends for the main macroeconomic indicators of the Russian economy.

Next steps are going to be concentrated at the analysis of the optimal switching time between two branches of the production function, corresponding to the periods before and after crucial changes in the economy; and the other part of the research is the application of different types of production functions, such as CES, LINEX, Cobb-Douglas and their combinations, for the original economic growth model.

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